TITLE: A STATISTICAL THEORY OF LANGMUIR TURBULENCE

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A STATISTICAL THEORY OF LANGMUIR TURBULENCE

by

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ABSTRACT

A statistical theory of Langmuir turbulence is developed by applying a generalization of the direction interaction approximation (DIA) of Kraichnan to the Zakharov equations describing Langmuir turbulence.

A STATISTICAL THEORY OF LANGMUIR TURBULENCE

Recent advances in the theory of "strong"
Langmuir turbulence[1] have concentrated on the evo-
lution of modulational instabilities and their
relation to soliton solutions (in 1-D) and Langmuir
cc apse[2] in higher dimensions. Most of the non-
linear theories in this area have emphasized initial
value or similarity solutions for the coherent non-
linear structures of interest. In many situations
we can conceive of the nonlinear excitations as
developing from the thermal (particle) fluctuations
in the plasma in the presence of external beams or
A.C. fields which drive Langmuir instabilities.
Numerical studies[3] have shown the birth and decay
of "collapsons", new collapsons being generated from
the residual fluctuation debris of previously
decayed collapsons. It is not unreasonable to try
to construct a statistical turbulence theory to
describe such situations. The present work repre-
sents a preliminary attempt in this direction.

Our method is a generalization to the Zakharov
equations[1] of the direct-interaction approximation
(DIA) of Kraichnan[4] which was originally applied to
the Navier-Stokes equations. We begin with the
Zakharov equations in Fourier transform space,
which can be written in the compact form:

$$L_{12}N_2 = \tfrac{1}{2}X^{(2)}_{123}N_2N_3 + \eta_1 , \tag{1}$$

where $N_1 = N_{\alpha 1}(\underline{k}_1,t_1)$ is a three component vector
with the discrete indices α_1 taking on the waves +,
0,-. Repeated indices such as 2 and 3 are under-
stood to be summed on discrete indices α_2, α_3 and
integrated over the continuous variables $\underline{k}_2$, t_2,
etc. The identification of the discrete indices is:

$$N_+(\underline{k}_1,t_1) = \phi^+(\underline{k}_1,t_1); \quad N_0(\underline{k}_1,t_1) = n(\underline{k}_1,t_1);$$
$$N_-(\underline{k}_1,t_1) = \phi^-(\underline{k}_1,t_1), \tag{2}$$

where $\phi^+(\underline{k}_1,t_1)$, $\phi^-(\underline{k}_1,t_1)$ are the spatial Fourier transforms of the envelope function of the high frequency Langmuir potential and its complex conjugate $\left[\phi^+(\underline{k},t)^* = \phi^-(-\underline{k},t)\right]$ and $n(k,t)$ is the Fourier transform of the ion density deviation $\left[n(\underline{k},t) = n(-\underline{k},t)\right]$. The linear operator L_{12} is

$$\begin{bmatrix} \partial_{t_1}+\nu_e+ik_1^2 & 0 & 0 \\ 0 & \partial_{t_1}^2+2\nu_i\partial_{t_1}+k_1^2 & 0 \\ 0 & 0 & \partial_{t_1}+\nu_e-ik_1^2 \end{bmatrix} \cdot (2\pi)^3\delta(\underline{k}_1-\underline{k}_2)\delta(t_1-t_2) \tag{3}$$

The only nonzero components of $X_{123}^{(2)}$ are

$$X_{+0-}^{(2)} = -X_{-0-}^{(2)} = -i(\underline{k}_1\cdot\underline{k}_3)k_1^{-2}\delta_{123} ,$$

$$X_{++0}^{(2)} = -X_{--0}^{(2)} = -i(\underline{k}_1\cdot\underline{k}_2)k_1^{-2}\delta_{123} ; \tag{4}$$

$$X_{0+-}^{(2)} = X_{0-+}^{(2)} = -k_1^2(\underline{k}_2\cdot\underline{k}_3)\delta_{123}$$

$$\delta_{123} = (2\pi)^3\delta^3(\underline{k}_1-\underline{k}_2-\underline{k}_3)\delta(t_2-t_3)\delta(t_1-t_2) .$$

Equation (1) is a general quadratically nonlinear form to which the DIA can be immediately applied,[5] giving a set of equations coupling three important objects: the ensemble averaged (mean) fields $\langle N_1\rangle$, the infinitesimal response function[5] (given by a functional derivative) and the covariance:

$$R_{11'} = \delta\langle N_1\rangle/\delta\langle\eta_1,\rangle; \quad C_{11'} = \langle N_1 N_1'\rangle-\langle N_1\rangle\langle N_1'\rangle \tag{5}$$

The equations connecting these are:

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$$(L_{12}-\Sigma_{12})R_{21'} = \delta_{11'}\left[=\delta_{\alpha1\alpha1'}\delta^3(\underline{k}_1-\underline{k}_1')\delta(t_1-t_1')\right],$$ (6)

where,

$$\Sigma_{11'} = X^{(2)}_{123}X^{(2)}_{2'3'1'}R_{22'}C_{33'}+X^{(2)}_{123}\langle N_3\rangle,$$ (7)

$$L_{12}\langle N_2\rangle = \langle \eta_1\rangle + \tfrac{1}{2}X^{(2)}_{123}(\langle N_2\rangle\langle N_3\rangle+C_{23}),$$ (8)

$$(L_{12}-\Sigma_{12})C_{21'} = R_{1'2'}S_{12'},$$ (9)

$$S_{11'} = S^0_{11'} + \tfrac{1}{2}X^{(2)}_{123}X^{(2)}_{1'2'3'}C_{22'}C_{33'},$$ (10)

where $S^c_{11'}$ is the correlation of the fluctuating part of the driving term η_1 which we can take, for example, to be white noise related to the damping decrements by a Nyquist theorem:

$$S^0_{11'} = \langle\delta\eta_1\delta\eta_{1'}\rangle = \Gamma_{\alpha1\alpha1'}\delta^3(k_1-k_1')\delta(t_1-t_1'),$$ (11)

$$\Gamma_{+-} = \Gamma_{-+} = 4\pi T_e\nu_e k_1^{-2}; \quad \Gamma_{00} = 3k_1^2(m_i/m_e)\nu_i;$$

others zero. (12)

This set of nonlinear, non-Markoffian coupled equations is currently being studied by numerical analysis. The general initial value problem is difficult because of the long time histories which must be retained in an unstably evolving system. The DIA can be shown to have the general property that the mean constants of the motion of the Zakharov equations[1,2] $\langle N\rangle$, $\langle\underline{p}\rangle$ and $\langle H\rangle$ are independent of time. These conservation properties of the complete DIA give us some confidence that these equations contain at least some of the physics important for Langmuir collapse.[2]

We have investigated the steady state, spatially homogeneous solutions of these equations

for zero mean field. Here the R(C) matrices have a diagonal (anti-diagonal) form. The explicit equations for the various components are:

$$\left[-i(\omega_1 - k_1^2) + \nu_e - \Sigma_{++}(k_1\omega_1)\right]R_{++}(k_1\omega_1) = 1 \qquad (13a)$$

$$\Sigma_{++}(k\omega) = (2\pi)^{-4}\int d^3k_2\int d\omega_2\{i(\underline{k}_1\cdot\underline{k}_2)^2 k_1^{-2}k_3^2$$

$$\cdot C_{+-}(\underline{k}_2,\omega_2)R_{00}(\underline{k}_3,\omega_3) + (\underline{k}_1\cdot\underline{k}_2)^2 k_1^{-2}k_2^{-2}$$

$$\cdot R_{++}(\underline{k}_2,\omega_2)C_{00}(\underline{k}_3\omega_3)\} \quad ; \qquad (13b)$$

$$\left[-i(\omega_1-k_1^2) + \nu_e - \Sigma_{++}(\underline{k}_1,\omega_1)\right]C_{+-}(\underline{k}_1\omega_1)$$

$$= R_{++}(k_1\omega_1)\, S_{+-}(\underline{k}_1,\omega_1) \qquad (14a)$$

$$S_{+-}(\underline{k}_1\omega_1) = S_{+-}^0(\underline{k}_1\omega_1) + (2\pi)^{-4}\int d^3k_1 \; d\omega_1 (\underline{k}_1\cdot\underline{k}_3)^2 k_1^{-4}$$

$$\cdot C_{00}(\underline{k}_2,\omega_2)C_{-+}(\underline{k}_3,\omega_3) \quad ; \qquad (14b)$$

$$\left[-\omega_1^2 + 2i\omega_1\nu_i + k_1^2 - \Sigma_{00}(\underline{k}_1,\omega_1)\right]R_{00}(\underline{k}_1,\omega_1) = 1, \quad (15a)$$

$$\Sigma_{00}(\underline{k}_1\omega_1) = (2\pi)^{-4}\int d^3k_2\int d\omega_2 (-i)(\underline{k}_2\cdot\underline{k}_3)^2 k_1^2 k_2^{-2}$$

$$\cdot\left[R_{++}(\underline{k}_2\omega_2)C_{-+}(\underline{k}_3\omega_3)-R_{--}(\underline{k}_2\omega_2)C_{+-}(\underline{k}_3\omega_3)\right]; \quad (15b)$$

$$\left[-\omega_1^2 + 2i\omega_1\nu_i + k_1^2 - \Sigma_{00}(\underline{k}_1,\omega_1)\right]C_{00}(\underline{k}_1,\omega_1)$$

$$= R_{00}(\underline{k}_1,\omega_1)\, S_{00}(\underline{k}_1,\omega_1) \; , \qquad (16a)$$

$$S_{00}(\underline{k}_1,\omega_1) = S_{00}^0(\underline{k}_1\omega_1) + (2\pi)^{-4}\int d^3k_1\int d\omega_1 k_1^4$$

$$\cdot(\underline{k}_2\cdot\underline{k}_3)^2 \; C_{+-}(\underline{k}_2\omega_2)C_{-+}(\underline{k}_3,\omega_3) \; ; \qquad (16b)$$

where throughout $\underline{k}_1 = \underline{k}_2 + \underline{k}_3$, $\omega_1 = \omega_2 + \omega_3$ and $R_{--}(\underline{k}_1\omega_1) = R_{++}(-\underline{k}_1,-\omega_1)^*$ and $C_{-+}(\underline{k}_1,\omega_1) = C_{+-}(-\underline{k}_1,-\omega_1)$.

What usually is described as weak turbulence theory is obtained from equations (14) and (16) by replacing the response functions R and everywhere by their zero order values $R_{++}^0 = \left[-i(\omega-k^2) + \nu_e\right]^{-1}$, $R_{00}^0 = \left[-\omega^2 + 2i\omega\nu_i + k^2\right]^{-1}$, $C_{+-}^0 = W(-\underline{k})k^{-2}(2\pi)\cdot\delta(\omega - k^2)$, etc., where $W(\underline{k})$ is the Langmuir energy

density. In this approximation, it is easily seen
that the two terms in Σ_{++} and Σ_{00} describe induced
decay processes of the form $L \to L+S$ which lead, for
example, to the well-known cascade of wave energy
toward lower k.

The DIA, on the other hand, treats the self-
consistent renormalization of the response functions
and this renormalization is necessary to preserve
the mean constants of the motion. This renormali-
zation has extremely important consequences: For
example, if in (15b) we approximate the quantities
R and C by their zero order values and then substi-
tute the resulting Σ_{00} into (15a) we find that
R_{00} has complex poles corresponding to the roots of
the well-known modulational dispersion relation,
i.e., the quantity in square brackets in (14) is
just this dispersion relation generalized to the
case of a broad Langmuir spectrum.[6] Thus, in
addition to the modified ion sound poles in R_{00}
(modified by the decay branch of the dispersion
relation) the modulational instability introduces
new modulational poles in R_{00}. When this renorma-
lization R_{00} is put into (13b), it appears that
flow of Langmuir energy toward higher k becomes
possible. Furthermore, C_{00} now has large compon-
ents due to the modulational instability which
produce a significant source term (14b) on the
right-hand side of (14a). The renormalized Langmuir
modes then have a _finite_ turbulent damping in con-
trast to the marginally damped modes found in the
usual theory of the cascade of energy toward small
k.

A fully self-consistent solution of the DIA equations requires computer analysis. We will present results of 1-D numerical solutions of the DIA equations as well as a comparison of the results with direct computer solutions of the Zakharov equations. Questions concerning the spontaneous generation of a mean field at $\underline{k} = 0$ to account for the Langmuir condensation and consequent modulational instability will also be addressed.

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